12 - Predicate logic (continued)

Free vs bound variable. A variable is bound if it is within the scope of a quantifier; otherwise it is free.

$$[\forall x \in \mathbb{Z} : x^2 \geq x + y + z] \wedge [\exists y \in \mathbb{R} : (\forall z \in \mathbb{Z} : x^2 \geq y + z)]$$

Quantify over x → All instances of x here are bound; the y and z are free variables.

Quantify over y

Quantify over z → The y and z are bound variables; x is free.

Convention. In common speech, a free variable is implicitly universally quantified:

Example. If $x \geq 1$, then $x^2 \leq x^3$. → For all $x \in \mathbb{R}$, if $x \geq 1$, then $x^2 \leq x^3$.

$$\forall x \in \mathbb{R} : [x \geq 1 \Rightarrow x^2 \leq x^3]$$

$$xy = 0 \Rightarrow x = 0 \text{ or } y = 0 \quad \longrightarrow \quad \forall (x, y) \in \mathbb{R}^2 : [xy = 0 \Rightarrow (x = 0) \vee (y = 0)]$$

Negating quantifiers. Intuitively $\forall$ and $\exists$ are supersized versions of $\bigwedge$ and $\bigvee$:

$$\forall x \in \{x_1, \ldots, x_n\} : P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots P(x_n)$$

$$\exists x \in \{x_1, \ldots, x_n\} : P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots P(x_n)$$

So $\neg$ distributes across terms and flips $\bigwedge$ and $\bigvee$ (de Morgan's law):

$$\neg[\forall x \in U : P(x)] \equiv \exists x \in U : P(x)$$

$$\neg[\exists x \in U : P(x)] \equiv \forall x \in U : P(x)$$

Quantifiers as loops.

$\forall x \in S : P(x)$ →

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for x in S:
    if not P(x) then
        return False
return True
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$\exists x \in S : Q(x)$ →

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for x in S:
    if Q(x) then
        return True
return False
```

Vacuous quantification.

$$\forall x \in \varnothing : P(x) \equiv T \qquad \exists x \in \varnothing : P(x) \equiv F$$

Optional Homework due April 1 or 2.
Show your work. Answer without work receives no credit.

1. True or False?
(a) $\exists x \in \mathbb{R} : x^2 > x$
(b) $\exists x \in \mathbb{R} : x^2 = -1$
(c) $\exists x \in \mathbb{R} : x^2 + 2 > 1$
(d) $\forall x \in \mathbb{N} : (x^2 \neq x) \vee (x = 0) \vee (x = 1)$
(e) $\exists n \in \mathbb{N} : n^2 \equiv 3 \pmod 4$

2. Explain why this proposition is false: $\forall x \in \mathbb{R} : x^2 \geq x$

3. Let C(x)="x has a cat", D(x)="x has a dog", F(x)="x has a ferret.", S={students in your class}. Formalize:
(a) A student in your class has a cat, a dog, and a ferret.
(b) All students in your class have a cat, a dog, or a ferret.
(c) Some student in your class has a cat and a ferret but not a dog.
(d) No student in this class has a cat, a dog, and a ferret.
(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has one of these animals.

13 - Nested quantifiers (Liben-Nowell 3.5)

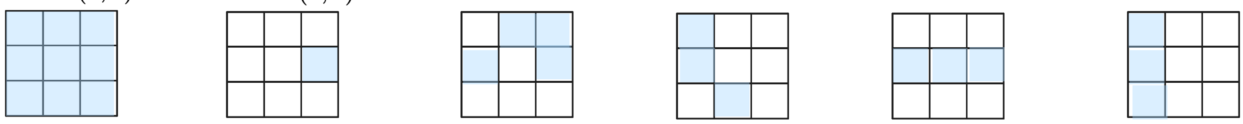
∀ commutes with ∀, ∃ commutes with ∃, but ∃ and ∀ do not commute (so order matters):

$$\forall(x,y) \in S \times T : P(x,y) \equiv \forall x \in S : [\forall y \in T : P(x,y)] \equiv \forall y \in T : [\forall x \in S : P(x,y)]$$

$$\exists(x,y) \in S \times T : P(x,y) \equiv \exists x \in S : [\exists y \in T : P(x,y)] \equiv \exists y \in T : [\exists x \in S : P(x,y)]$$

Let P(r,c) = "Row r column c is filled in the grid."

$\forall r \forall c\ P(r,c) \equiv \forall c \forall r\ P(r,c)$ | $\exists r \exists c\ P(r,c) \equiv \exists c \exists r\ P(r,c)$ | $\forall c \exists r\ P(r,c)$ | $\forall r \exists c\ P(r,c)$ | $\exists r \forall c\ P(r,c)$ | $\exists c \forall r\ P(r,c)$



Negating nested quantifiers: "It is not the case that every grid cell is filled."

$$\equiv \neg[\forall r : [\forall c : P(r,c)]] \equiv \neg[\exists r : \exists c : \neg P(r,c)]$$

= "There exists an empty grid cell."

Translations:

1. A reciprocal of a real number x is another real number y such that xy=1.
   "Every nonzero real number has a reciprocal."

$$\forall x \in \mathbb{R} - \{0\} : [\exists y \in \mathbb{R} : xy = 1]$$

$$\forall x \in \mathbb{R} : [x \neq 0 \Rightarrow (\exists y \in \mathbb{R} : xy = 1)]$$

$$\forall x \in \mathbb{R} : [(x = 0) \vee (\exists y \in \mathbb{R} : xy = 1)]$$

2. "There is a smallest natural number."

$$\exists n \in \mathbb{N} : [\forall m \in \mathbb{N} : n \leq m]$$

3. "There is no smallest positive real number."

   = NOT("There is a smallest positive real number.")

$$\neg[\exists n \in \mathbb{R} : [(n > 0) \wedge (\forall m \in \mathbb{R} : (m \leq 0) \vee (n \leq m))]]$$

$$\equiv \forall n \in \mathbb{R} : [(n \leq 0) \vee (\exists m \in \mathbb{R} : (m > 0) \wedge (n > m))]$$

4. "Every value in the sequence (x_1, x_2, x_3, x_4) appears more than once."
   Examples: (2,2,1,1), (1,2,3,3), (-6,3,2,3).

$$\forall i \in \{1,2,3,4\} : [\exists j \in \{1,2,3,4\} : (i \neq j) \wedge (x_i = x_j)]$$

Optional Homework due April 8 or 9.
Show your work. Answer without work receives no credit.

1. True or False?
(a) $\exists x \in \mathbb{R} : [\forall y \in \mathbb{R} : xy = 0]$
(b) $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : x = y^2]$
(c) $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : x^2 = y]$
(d) $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : (xy > 0) \vee (x = 0)]$
(e) $\exists x \in \mathbb{R} : [\forall y \in \mathbb{R} : (xy > 0) \vee (y = 0)]$
(f) $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : (x \neq 0) \Rightarrow (xy > 0)]$

2. Formalize the statement that any two entries of the sequence (x_1, x_2, x_3) are distinct.

3. A sequence of real numbers $(x_1, x_2, x_3, x_4, \ldots)$ is bounded if there exists an M > 0 such that $|x_i| \leq M$ for all $i \in \mathbb{N}$.
(a) Use predicate logic to formalize the notion of a bounded sequence.
(b) Negate the logical statement in (a) so that the resulting proposition does not use the symbol ¬.